Fluid Dynamics

Non-Ideal Fluids

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Introduction

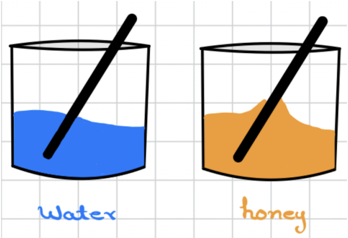
In this chapter, we will define **non-ideal fluids** and study the properties that arise from **viscosity**. Our study will be limited to **incompressible fluids**.

We will focus on the following topics:

- The Navier–Stokes equation
- Momentum Equation
- Dimensional Analysis
- Reynolds Number

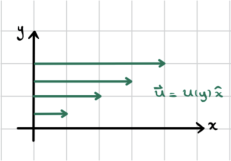
Viscosity

Viscosity is the property of a fluid that quantifies its resistance to deformation due to velocity gradients between adjacent fluid layers. It acts to reduce these differences by exerting internal friction, thereby smoothing out velocity variations over time.



Viscosity Coefficient η

Fluids with a constant viscosity coefficient η, regardless of the applied shear, are called **Newtonian fluids**.



$$\frac{\text{Force}}{\text{Area}} = \eta \cdot \frac{\partial u_x}{\partial y}$$

The **kinematic viscosity** ν is defined as the ratio of dynamic viscosity to density:

$$\nu = \frac{\eta}{\rho}$$

Viscous force density

We observe that the **viscous force** acts parallel to the surface — it corresponds to a **shear stress**. In contrast, **pressure** is associated with the microscopic collisions of particles perpendicular to the surface, representing a **normal stress**.

The upper fluid layers, which move at higher velocities, exert shear forces that tend to entrain the slower-moving lower layers beneath them.

Therefore, the presence of viscosity ensures the existence of a force component parallel to the surface, known as the **viscous force density**:

$$\frac{\text{Force}}{\text{Volume}} = \eta \cdot \vec{\nabla}^2 \vec{u}$$

Navier–Stokes Equation

The Navier–Stokes equations represent the **momentum equation** for a fluid, including the **viscous force density**.

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) = -\vec{\nabla}P + \vec{f} + \eta\,\vec{\nabla}^2\vec{u}$$

- acceleration
- force per unit volume due to pressure
- force per unit volume due to external forces
- force per unit volume due to viscosity

Momentum Equation

The momentum equation is written as:

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\mathbf{\Pi} = \vec{f} \quad \rightarrow \quad \frac{\partial}{\partial t}(\rho u_i) + \sum_{j=1}^{3}\frac{\partial \mathbf{\Pi}_{ij}}{\partial x_j} = f_i$$

In the momentum flux tensor $\mathbf{\Pi}_{ij}$, the effect of viscosity must be included:

$$\mathbf{\Pi}_{ij} = \rho u_i u_j - \sigma_{ij}$$

where the **stress tensor** σ_{ij} is:

$$\sigma_{ij} = -P\delta_{ij} + \sigma'_{ij}$$

and the **viscous stress tensor** is:

$$\sigma'_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \rightarrow \quad \textbf{viscous contribution in Cartesian coordinates}$$

Dimensional Analysis

The Navier–Stokes equation for an incompressible, steady flow without external forces is:

$$\rho\left(\cancel{\frac{\partial\vec{u}}{\partial t}}+(\vec{u}\cdot\vec{\nabla})\vec{u}\right)=-\vec{\nabla}P+\cancel{\vec{f}}+\eta\,\vec{\nabla}^2\vec{u}$$

Define the following dimensionless variables:

$$\vec{r}=L\vec{r}^*,\quad \vec{\nabla}=\frac{1}{L}\vec{\nabla}^*,\quad \vec{u}=U\vec{u}^*,\quad P=\rho U^2P^*$$

Substituting into the Navier–Stokes equation and simplifying leads to the dimensionless form:

$$(\vec{u}^*\cdot\vec{\nabla}^*)\vec{u}^*=-\vec{\nabla}^*P^*+\frac{1}{Re}\vec{\nabla}^{*2}\vec{u}^*,\quad \text{where}\quad Re=\frac{\rho UL}{\eta}=\frac{UL}{\nu}\quad \text{(Reynolds number)}$$

Dimensional Analysis

Therefore, if two different flows of the same type (e.g., flow around a sphere) have the same Reynolds number R_e, then knowing the solution for one flow allows us to determine the solution for the other.

- Characteristic velocity (e.g., velocity far from the object)
- Characteristic length (e.g., size of the object or boundary layer thickness)

Reynolds Number

The Reynolds number is defined as:

$$Re = \frac{\text{Inertia}}{\text{Viscosity}} = \frac{\rho U L}{\eta} = \frac{UL}{\nu}$$

Note that dimensionless numbers do not remove the physical meaning of our problems but help us identify which terms dominate the flow behavior.

- $Re \gg 1$: Viscosity can be neglected (inertia dominates)

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} P$$

- $Re \ll 1$: Inertia can be neglected (viscosity dominates)

$$-\vec{\nabla} P + \eta \, \vec{\nabla}^2 \vec{u} = 0$$

Stokes Drag

Stokes drag develops at the boundary between the fluid and the surface, where viscosity plays a crucial role:

- For $Re \ll 1$: $\vec{F} \propto \eta UL$ (Stokes law), e.g. $F = 6\pi\eta RU$ for a sphere
- For $Re \gg 1$: $\vec{F} \propto \frac{1}{2} C_D \rho S U^2$ (drag force, with $\vec{F}$ opposite to the flow direction)

The total drag force consists of contributions from both pressure and viscous forces.

Magnus Effect

The **Magnus effect** refers to the lateral force experienced by a spinning object moving through a fluid, causing it to deviate from a straight-line path due to asymmetric pressure distribution.

1. Spinning Cylinder (2D – Analytical):

$$F_M = \rho U \Gamma, \quad \Gamma = 2\pi R^2 \omega$$

2. Spinning Ball (3D – Empirical):

$$F_M = C_L \cdot \frac{1}{2}\rho U^2 A, \quad A = \pi R^2$$

with spin parameter $S = \frac{\omega R}{U}$ and $C_L \approx 1.2S$ for small S.

Magnus Effect