Fluid Dynamics

Stream Function Ψ

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Introduction

In this chapter, we will study fundamental properties that arise from the **velocity field** of a fluid, specifically:

- **Vorticity** $\vec{\zeta}$
- **Circulation** Γ
- **Velocity potential** Φ
- **Stream function** Ψ

Vorticity $\vec{\zeta}$

Vorticity describes the rotational tendency of the velocity field $\vec{u}$ and is given by:

$$\vec{\zeta} = \vec{\nabla} \times \vec{u}$$

In practice, vorticity is twice the local angular velocity:

$$\vec{\zeta} = 2 \cdot \vec{\omega} \quad \Rightarrow \quad \vec{u} = \vec{\omega} \times \vec{r}$$

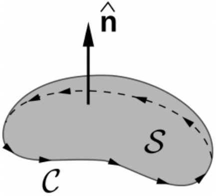
In cases where a velocity field has no vorticity, but an angular velocity still exists, the flow is called an **irrotational vortex** (or free vortex).

An intuitive way to understand vorticity is to imagine placing a small paddle wheel at a point of interest and observing how it spins due to the local velocity gradients.

Circulation Γ

The circulation Γ of the velocity field $\vec{u}$ around a closed curve C is equal to the flux of vorticity $\vec{\zeta}$ through any surface S bounded by C:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{r} = \iint_S \vec{\zeta} \cdot d\vec{a}$$



Kelvin's Theorem

The circulation around a closed curve moving with the fluid remains constant over time:

$$\frac{d\Gamma}{dt} = 0 \quad \Rightarrow \quad \text{Circulation is conserved}$$

When can we apply this? In cases where there is no viscosity and the forces are conservative (e.g., no magnetic forces), such that

$$\frac{\nabla P}{\rho} = \nabla h \quad \Rightarrow \quad \text{Kelvin's criterion}$$

where h depends on the fluid type:

- Incompressible fluid: $h = \frac{P}{\rho}$
- Adiabatic, steady flow: $h =$ specific enthalpy
- Adiabatic, steady ideal gas: $h = \frac{\gamma}{\gamma-1}\frac{P}{\rho}$
- Isothermal: $h = \frac{k_B T}{m} \ln(\rho)$

Kelvin's Theorem

Using the above theorem, the evolution of vorticity can be expressed as:

$$\frac{d\vec{\zeta}}{dt} = (\vec{\zeta} \cdot \nabla)\vec{u} - \vec{\zeta}(\nabla \cdot \vec{u})$$

Note that for an incompressible fluid:

$$\nabla \cdot \vec{u} = 0 \quad \Rightarrow \quad \vec{\zeta}(\nabla \cdot \vec{u}) = 0$$

An important consequence of the vorticity equation is that in regions of the fluid where the velocity field is irrotational, the vorticity vanishes:

$$\vec{\zeta} = 0$$

Velocity Potential Φ

In the case where the velocity field is irrotational, $\vec{\zeta} = 0$, we require the existence of a scalar potential such that:

$$\vec{\nabla} \times \vec{u} = 0 \quad \Rightarrow \quad \vec{u} = -\vec{\nabla}\Phi$$

- **Cartesian Coordinates:**

$$u_x = -\frac{\partial \Phi}{\partial x}, \quad u_y = -\frac{\partial \Phi}{\partial y}, \quad u_z = -\frac{\partial \Phi}{\partial z}$$

- **Polar Coordinates:**

$$u_r = -\frac{\partial \Phi}{\partial r}, \quad u_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$

Velocity Potential Φ

For an **irrotational** fluid, the momentum equation reads:

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) + \vec{\zeta} \times \vec{u} = -\nabla(h + \Phi_g)$$

Setting $\vec{\zeta} = 0$ and $\vec{u} = -\nabla\Phi$, we get:

$$-\frac{\partial \Phi}{\partial t} + \frac{|\nabla\Phi|^2}{2} + h + \Phi_g = f(t)$$

which is the Bernoulli equation for irrotational flow, where

$$f(t) = E \quad \forall \text{ streamline.}$$

Therefore, to find the pressure P, we consider:

- $\vec{\zeta} = 0$: from Bernoulli equation
- $\vec{\zeta} \neq 0$: from momentum equation

Stream Function Ψ

For an incompressible fluid, we start from the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{u} = 0$$

Therefore, there exists a **stream function $\boldsymbol{\Psi}$** $\in \mathbb{R}^2$ such that

$$\vec{u} = \nabla \times (\Psi \hat{z}) = \nabla \Psi \times \hat{z}$$

Knowing the expression for the stream function Ψ allows us to determine the streamlines, which correspond to

$$\Psi = \text{constant.}$$

Stream Function Ψ

- **Cartesian coordinates:**

$$u_x = \frac{\partial \Psi}{\partial y}, \quad u_y = -\frac{\partial \Psi}{\partial x}$$

- **Polar coordinates:**

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \Psi}{\partial r}$$

- **Axisymmetric cylindrical coordinates:**

$$u_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}$$

Laplace Equation

For an irrotational flow, we seek a velocity potential Φ, while for an incompressible velocity field, we seek a stream function Ψ. Consider a fluid with the following properties:

- Incompressible and irrotational: $\nabla \cdot \vec{u} = 0 \quad \Rightarrow \quad \nabla^2 \Phi = 0$
- Incompressible and irrotational (2D): $\nabla \times \vec{u} = 0 \quad \Rightarrow \quad \nabla^2 \Psi = 0$

$$\nabla \Phi \cdot \nabla \Psi = 0$$

The gradients of Φ and Ψ are perpendicular, so equipotential lines and streamlines intersect at right angles.

Boundary Conditions

Boundary conditions are essential to uniquely solve flow problems. Common types include:

- **No-slip condition:** Velocity of the fluid at a solid boundary equals the velocity of the boundary itself,

$$\vec{u} = \vec{u}_{\text{wall}}$$

- **Free-slip (impermeable) condition:** No penetration through the boundary (normal velocity zero),

$$\vec{u} \cdot \hat{n} = 0 \quad \Rightarrow \quad \frac{\partial \Phi}{\partial \hat{n}} = 0$$

- **Inflow/Outflow condition:** Continuity of normal derivatives of velocity potential across the boundary,

$$\vec{u} = \vec{u}_{\text{inflow}}, \quad p = p_{\text{outflow}} \quad \Rightarrow \quad \frac{\partial \phi_{\text{out}}}{\partial n} - \frac{\partial \phi_{\text{in}}}{\partial n} = 0$$

Laplace Equation - Solutions

The well-known solutions to Laplace's equation are:

- **Cylindrical Coordinates:**

$$\Phi(\rho, \theta) = A_0 + B_0 \ln(\rho) + \sum_{m=1}^{\infty} \left(A_m \rho^m + \frac{B_m}{\rho^m} \right) \cos(m\theta) + \sum_{m=1}^{\infty} \left(C_m \rho^m + \frac{D_m}{\rho^m} \right) \sin(m\theta)$$

- **Spherical Coordinates:**

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Complex Potential

For incompressible and irrotational flows, we can define the **complex potential** $W(z)$, where:

$$\begin{cases} \vec{u} = \vec{\nabla}\Phi \\ \vec{u} = \vec{\nabla} \times \Psi\,\hat{z} \end{cases} \qquad \Rightarrow \qquad W(z) = \Phi(x, y) + i\,\Psi(x, y)$$

The functions Φ and Ψ satisfy the Cauchy–Riemann equations:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

Thus, $W(z)$ is an analytic function of the complex variable $z = x + iy$, and contains the full information about the velocity field.

Characteristic Flow Sources

The mass flow rate is defined as:

$$\dot{M} = \iint_A \rho\, \vec{u} \cdot d\vec{A}$$

- **Spherical Source – $\dot{M}$:**

$$u_r = \frac{1}{4\pi r^2} \cdot \frac{\dot{M}}{\rho}, \quad \Phi(r) = -\frac{1}{4\pi r} \cdot \frac{\dot{M}}{\rho}$$

- **Line Source – $\dot{M}/L$:**

$$u_\varpi = \frac{1}{2\pi\varpi} \cdot \frac{\dot{M}}{\rho L}, \quad \Phi(\varpi) = \frac{\dot{M}}{2\pi\rho L} \ln(\varpi)$$

Note: The volume flow rate $\frac{\dot{M}}{\rho}$ corresponds to the electric charge q in electrostatics!

Coandă Effect

As fluid flows along a curved surface of radius R, the pressure gradient normal to the flow is related to the centrifugal acceleration by

$$\frac{u^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial n}.$$

Because pressure decreases toward the center of curvature, the low-pressure region "pulls" the fluid, keeping it attached to the surface.

Conclusion: $\dfrac{\partial P}{\partial n} < 0$ for positive curvature, so the flow follows the surface.