Fluid Dynamics

Conservation Laws II

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Introduction

In this chapter, we study ideal fluids and the fundamental **conservation laws**, as before, but now we aim to express them in a **continuity form**:

$$\frac{\partial}{\partial t}(\text{density}) + \vec{\nabla} \cdot (\text{flow}) = \text{sources} - \text{sinks}$$

Energy Equation

To formulate the energy equation in its most general form, we must compute the following quantities:

- **Energy density**:

$$\delta E = \delta m \cdot \frac{u^2}{2} + \delta m \cdot \varepsilon \quad \Rightarrow \quad \int_V \left(\rho \cdot \frac{u^2}{2} + \rho \cdot e \right) d\tau \quad \text{(Total energy per unit volume)}$$

- **Energy flux***:

$$\oint_{\partial V} \left(\rho \cdot \frac{u^2}{2} + \rho \cdot e \right) \vec{u} \cdot d\vec{a} \cdot dt \quad \text{(Convective energy flux)}$$

Energy Equation – Source and Sink Terms

- **Energy sources**:

$$\int_V \rho \cdot q \, d\tau \cdot dt \quad \text{(Volumetric heating or cooling)}$$

$$\int_V \vec{f} \cdot \vec{u} \, d\tau \cdot dt \quad \text{(Work done by body forces)}$$

- **Energy sinks***:

$$\oint_{\partial V} P \, \vec{u} \cdot d\vec{a} \cdot dt \quad \text{(Work done by pressure forces on the surroundings)}$$

Energy Equation

The equations marked with * represent closed surface integrals and are summarized as follows:

$$\oint_{\partial V} \left(\rho \cdot \frac{u^2}{2} + \rho \cdot e + P \right) \vec{u} \cdot d\vec{a} \cdot dt$$
$$= \oint_{\partial V} \rho \left(\frac{u^2}{2} + h \right) \vec{u} \cdot d\vec{a} \cdot dt$$

where:

$$h = e + \frac{P}{\rho} \quad \text{(specific enthalpy)}$$

Energy Equation

Therefore,

$$\frac{\partial}{\partial t}\left(\rho \cdot \frac{u^2}{2} + \rho \cdot e\right) + \vec{\nabla} \cdot \left(\rho \cdot \frac{u^2}{2} \cdot \vec{u} + \rho \cdot h \cdot \vec{u}\right) = \vec{f} \cdot \vec{u} + pq$$

where:

$$pq > 0 \quad \Rightarrow \quad \textbf{Heating}$$

$$pq < 0 \quad \Rightarrow \quad \textbf{Cooling}$$

Momentum Equation

Following the same procedure as before, the conservation of momentum for an ideal fluid can be written in continuity form as:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + f_i$$

where:

- ρu_i: **momentum density** in the *i*-th direction
- $\frac{\partial}{\partial x_j}(\rho u_i u_j)$: **macroscopic momentum flux** (convective transport)
- $\frac{\partial P}{\partial x_i}$: **microscopic momentum flux** due to pressure gradients
- f_i: external body force per unit volume (e.g., gravity)