Fluid Dynamics

Conservation Laws I

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July 4, 2025

Introduction

In this chapter, we study ideal fluids — fluids that exhibit no viscosity. The discussion applies to both compressible and incompressible flows. Specifically, we examine three fundamental conservation laws:

- Conservation of Mass (Continuity Equation)
- Conservation of Momentum
- Conservation of Energy

Conservation of Mass

The conservation of mass is expressed by the continuity equation, which in general form reads:

$$\frac{\partial(\textit{density})}{\partial t} + \vec{\nabla} \cdot (\textit{flow}) = \textit{sources} - \textit{sinks}$$

For a fluid with density ρ and velocity field $\vec{u}$, this becomes:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

Note: If mass sources or sinks are present (e.g., chemical reactions or mass injection), the right-hand side is non-zero.
In the case of **incompressible flow** ($\rho =$ constant), the continuity equation simplifies to:

$$\vec{\nabla} \cdot \vec{u} = 0$$

Conservation of Mass

The divergence $\vec{\nabla} \cdot \vec{u}$ describes the local rate of volumetric expansion or compression of the fluid:

- $\vec{\nabla} \cdot \vec{u} > 0$: the flow diverges — the fluid expands locally (acts as a source),
- $\vec{\nabla} \cdot \vec{u} < 0$: the flow converges — the fluid compresses locally (acts as a sink).

The conservation of mass can also be written in integral form:

$$-\iiint_V \frac{\partial \rho}{\partial t}\, d\tau = \iint_{\partial V} \rho \vec{u} \cdot d\vec{a}$$

Left-hand side: rate of increase of mass inside the control volume V.
Right-hand side: net mass flux *out* of the control volume across the surface ∂V.

Conservation of Momentum

The conservation of momentum in a fluid expresses Newton's second law: the rate of change of momentum equals the sum of forces.
For an inviscid fluid, the Euler equation is:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right) = -\vec{\nabla} P + \vec{f_b}$$

Where:

- Eulerian acceleration — total acceleration of a fluid particle,
- pressure gradient force — acts from high to low pressure (per unit volume),
- body force density — external force per unit volume acting throughout the fluid (e.g., gravity: $\vec{f_b} = \rho \vec{g}$).

Conservation of Momentum

From the conservation of momentum, we can compute the **total force exerted by pressure** on a control volume:

$$\vec{F}_{\text{pressure}} = -\iint_{\partial V} P \, d\vec{a}$$

This surface integral represents the net pressure force acting on the boundary ∂V.

Using the divergence theorem:

$$\vec{F}_{\text{pressure}} = -\iiint_V \vec{\nabla} P \, d\tau, \quad \text{since} \quad \iint_{\partial V} P \, d\vec{a} = \iiint_V \vec{\nabla} P \, d\tau$$

Tip: For any closed surface, $\iint_{\partial V} d\vec{a} = 0$. This allows us to replace a complex surface with a simpler one to ease pressure force calculations, as long as the total surface remains closed.

Conservation of Energy

The conservation of energy for compressible fluids (e.g., gas) is:

$$\delta Q = dU + P\,d\tau \quad \Rightarrow \quad \delta q = de + P\,d\tau$$

while:

$$\delta q = de + P\,d\tau \quad \text{with} \quad e = \frac{1}{\gamma - 1}\frac{P}{\rho}$$

The conservation of energy leads to the following expressions for specific thermodynamic processes:

- **Adiabatic process:** $\delta q = 0 \quad \Rightarrow \quad \dfrac{P}{\rho^\gamma} = \text{const}$
- **Isothermal process:** $T = \text{const} \quad \Rightarrow \quad \dfrac{P}{\rho} = \text{const}$
- **Polytropic gas:** $\gamma = \frac{C_p}{C_v} \quad \Rightarrow \quad P = A\rho^{\gamma_{\text{eff}}}$
- **Ideal (isentropic) fluid:** $\quad \Rightarrow \quad \dfrac{d}{dt}\left(\dfrac{P}{\rho^\gamma}\right) = 0$

Boundary Conditions

To solve the governing PDEs, we must define the appropriate boundary conditions:

- $\vec{u}_{\perp} = \vec{u}_{\text{surface}}$ (normal velocity must match the boundary velocity),
- $P = P_{\text{surface}}$ (pressure continuity for ideal fluids).

For static fluids ($\vec{u} = 0$), the boundary conditions reduce to:

- $\vec{u}_1 = \vec{u}_2$ (velocity continuity across the interface),
- $P_1 = P_2$ (pressure continuity across the interface for ideal fluids).

Bernoulli's Equation

Bernoulli's equation is derived by requiring the flow to be steady and inviscid. From the momentum equation:

$$\vec{u} \cdot \vec{\nabla} \left(\frac{u^2}{2} + h + \Phi_g \right) = 0 \quad \Rightarrow \quad \frac{u^2}{2} + h + \Phi_g = E$$

where E is constant along each streamline.

Note that Bernoulli's equation holds only if:

$$\vec{u} \cdot \frac{\vec{\nabla} P}{\rho} = \vec{u} \cdot \vec{\nabla} h$$

Special forms:

- **Incompressible fluids:** $h = \frac{P}{\rho} \quad \Rightarrow \quad \frac{u^2}{2} + \frac{P}{\rho} + \Phi_g = E$
- **Ideal gas (isentropic):** $h = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \quad \Rightarrow \quad \frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P}{\rho} + \Phi_g = E$

Acceleration and Pressure

The dependence of pressure on acceleration is given by the momentum conservation equation:

$$\rho \vec{a} = -\vec{\nabla} P + \cancelto{0}{\vec{f}}$$

This simplified form is valid only when external body forces (such as gravity) are negligible.

Note: Although $\vec{\nabla} \times \vec{\nabla} P = 0$ holds identically, the pressure gradient alone cannot generate rotational motion in the fluid. If the flow is rotational, it must be due to other forces (e.g., body forces or viscosity).