Fluid Dynamics

Introduction to Fluid Dynamics

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Introduction

These notes provide a survey of basic concepts in fluid dynamics as a preliminary to the study of Quantum Computational Fluid Dynamics. In this first chapter, we will cover:

- Fluid kinematics
- Types and characteristics of fluid flow
- Simplified flow equations

Lagrangian Description of Fluid Motion

In the **Lagrangian framework**, we describe fluid motion by following individual fluid particles through space and time. Each particle is labeled by its initial position $\vec{x}_0$ at time $t = 0$. The particle trajectory is given by the function:

$$\vec{x} = \vec{x}(\vec{x}_0, t)$$

This approach is analogous to classical mechanics, where we solve Newton's equations with specified initial conditions. The velocity and acceleration of a fluid particle are obtained by time derivatives:

$$\vec{u} = \frac{d\vec{x}}{dt}, \quad \vec{a} = \frac{d^2\vec{x}}{dt^2}$$

Eulerian Description of Fluid Motion

In the **Eulerian framework**, we describe fluid motion by observing how flow properties like velocity and pressure vary at fixed spatial points over time. Instead of following particles, we examine fields such as:

$$\vec{u} = \vec{u}(\vec{x}, t)$$

To capture the total change experienced by a moving fluid element, we use the **material derivative**:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F$$

This combines local (temporal) and convective (spatial) changes, and expresses evolution without reference to initial particle positions.

Flow Visualization Tools

To describe and visualize fluid motion, we use three fundamental tools:

- **Pathlines**
- **Streamlines**
- **Streaklines**

Each provides a different perspective on the flow and will be defined in the following slides.

Flow Visualization Tools

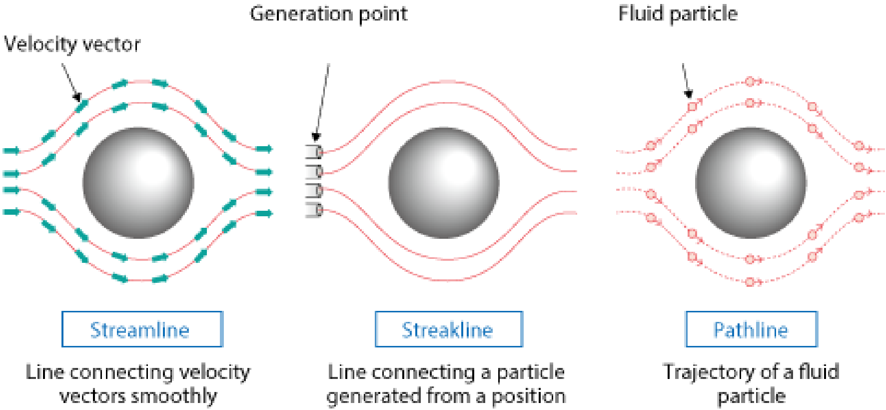


Figure: Streamline, streakline, and pathline in steady flow [**1**].

Flow Visualization Tools – Pathlines

Pathlines represent the trajectories traced by individual fluid particles as they move through the flow field. Their evolution is governed by the initial value problem:

$$\begin{cases} \dfrac{d\vec{x}_p(t)}{dt} = \vec{u}(\vec{x}_p(t), t) \\ \vec{x}_p(t_0) = \vec{x}_{p0} \end{cases}$$

To obtain the explicit form of the pathline, this system must be solved to determine the pathline.

Flow Visualization Tools – Streamlines

Streamlines are spatial curves that are everywhere tangent to the velocity field at a fixed moment in time. They represent the instantaneous direction of fluid motion.
The condition for a streamline is that its tangent vector is parallel to the velocity vector:

$$\frac{d\vec{x}_s}{ds} \times \vec{u}(\vec{x}_s) = 0$$

This leads to the classical scalar form:

$$\frac{dx_s}{u_x} = \frac{dy_s}{u_y} = \frac{dz_s}{u_z}$$

Streamlines are determined by solving this system at a fixed time $t = t_0$, and they provide a snapshot of the flow field's spatial structure.

Flow Visualization Tools – Streaklines

Streaklines are the loci of all fluid particles that have passed through a fixed spatial point over time. They are commonly visualized in experiments by continuously releasing dye or smoke from a single point.

To construct a streakline, we track multiple pathlines of particles that all passed through the same fixed point $\vec{x}_0$ at different times $\tau \leq t$. For each such particle, we solve:

$$\begin{cases} \dfrac{d\vec{x}_p(\tau)}{d\tau} = \vec{u}(\vec{x}_p(\tau), \tau) \\ \vec{x}_p(\tau = t') = \vec{x}_0, \quad t' \leq t \end{cases}$$

The streakline at time t is formed by connecting all such positions $\vec{x}_p(t)$.

Flow Classifications and Mathematical Criteria

Key types of fluid flow and their defining conditions:

- **Static flow:** No motion at any point in the domain:

$$\vec{u} = 0$$

- **Steady flow:** All flow properties remain constant over time:

$$\frac{\partial u}{\partial t} = \frac{\partial \rho}{\partial t} = \frac{\partial P}{\partial t} = 0$$

* In steady flows, streamlines, pathlines, and streaklines are identical.

- **Incompressible flow:** Fluid density is constant in space and time (volume-preserving):

$$\nabla \cdot \vec{u} = 0$$

- **Irrotational flow:** No local rotation or vorticity:

$$\nabla \times \vec{u} = 0$$

Simplified Flow Equations

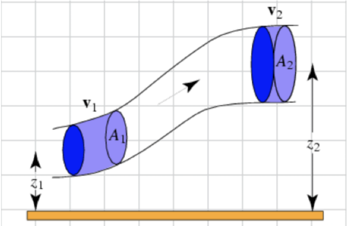
Two basic equations governing **non-static, steady flow** are the Conservation of Mass and the Bernoulli's equetion.

The **Conservation of Mass** or the Continuity Equation is,

$$pA_1 u_1 \, dt = pA_2 u_2 \, dt$$

$$\text{Mass flux:} \quad \frac{\dot{m}}{A} = \rho u \quad [\mathrm{kg/m^2 \cdot s}]$$

$$\text{Volumetric flow rate:} \quad Q = Au \quad [\mathrm{m^3/s}]$$



Simplified Flow Equations

Bernoulli's equation expresses conservation of mechanical energy *along a streamline* (for inviscid, incompressible, steady flow).

Starting from the mechanical energy of a fluid element:

$$\text{Potential energy:} \quad \Delta U = mg(z_2 - z_1)$$

$$\text{Kinetic energy:} \quad \Delta K = \frac{1}{2}m(u_2^2 - u_1^2)$$

$$\text{Work by pressure forces:} \quad W = (P_1 - P_2)V$$

Dividing all terms by mass $m = \rho V$, we obtain the Bernoulli equation:

$$\frac{P}{\rho} + \frac{u^2}{2} + gz = \text{constant}$$

For a perfect gas (isentropic, compressible flow), Bernoulli becomes:

$$\frac{\gamma}{\gamma - 1} \cdot \frac{P}{\rho} + \frac{u^2}{2} + gz = \text{constant}$$

Hydrostatic Equation

Hydrostatic equation describes how pressure varies with depth in a **static**, **incompressible** fluid.

Starting from vertical force balance on a fluid element:

$$\text{Upward pressure force:} \quad dp \cdot A$$

$$\text{Downward weight:} \quad \rho g A\, dz$$

$$\Rightarrow \quad dp = -\rho g\, dz \quad \Rightarrow \quad \frac{dp}{dz} = -\rho g$$

Integrating (for constant ρ):

$$P(z) = P_0 - \rho g(z - z_0)$$

This equation is widely used in fluid statics, atmospheric science, and engineering, where the fluid can be assumed at rest and density constant.

References I

[1] https://www.cradle-cfd.com/media/column/a147